Rarita–Schwinger Fields on Nearly Parallel G_2 Manifolds

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September 2023

1 Introduction

These notes will overview results from [Ohn22] regarding the spaces of Rarita–Schwinger fields on nearly parallel G_2 manifolds. Much of the work done there involve general gradients and standard Laplacian operators (see [Hom16, SW19]). We take a more tensorial approach to arrive at the same conclusions, though the underlying methodology is similar (squaring the Dirac operator and solving an appropriate system of equations). The author hopes that the tensorial identities may be of use in the future.

All structures invovled will be smooth unless stated otherwise. On a Riemannian manifold M, we use the metric g to identify vector fields and 1-forms. Tensor calculations will be done pointwise with respect to a local orthonormal frame $\{e_1, \dots, e_n\}$ such that $\nabla_i e_j = 0$ at the center. As such, all indices involved will be subscripts. We employ the Einstein summation convention throughout, so repeated subscripts are summed over the values 1 to $\dim M$.

Our convention for the Riemann curvature tensor is

$$\mathrm{Rm}(X, Y, Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle. \tag{1.1}$$

2 The Twisted Dirac and Rarita–Schwinger Operators

Let M be a Riemannian spin n-fold with metric g. The spinor bundle $\mathbb{S}$ of M is a Dirac bundle and we can define a Dirac operator $D_0 \colon \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ on it. With respect to a local orthonormal frame, D_0 acts on a spinor ϕ by

$$D_0 \phi = \sum_i e_i \cdot \nabla^S_i \phi. \tag{2.1}$$

The Dirac operator D_0 is self-adjoint with respect to the L^2-inner product.

We can embed $\mathbb{S}$ into the bundle $T^*M \otimes \mathbb{S}$ of spinor-valued 1-forms via the map ι defined by

$$\iota(\phi) = -\frac{1}{n} \sum_i e_i \otimes [e_i \cdot \phi], \tag{2.2}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame. The map ι has a left inverse given by Clifford multiplication μ, where

$$\mu(X \otimes \phi) = X \cdot \phi. \tag{2.3}$$

Using these two maps, we obtain a decomposition of the bundle $T^*M \otimes \mathbb{S}$ into

$$T^*M \otimes \mathbb{S} = \mathbb{S}_{\frac{1}{2}} \oplus \mathbb{S}_{\frac{3}{2}}, \tag{2.4}$$

where we identify $\mathbb{S}_{\frac{1}{2}} = \mathbb{S}$ with its image under ι and set $\mathbb{S}_{\frac{3}{2}} = \ker \mu$. We refer to objects in these spaces $\frac{1}{2}$-spinors and $\frac{3}{2}$-spinors respectively and let $\mathrm{pr}_{\frac{1}{2}} = \iota \circ \mu$ and $\mathrm{pr}_{\frac{3}{2}} = \mathrm{id} - \mathrm{pr}_{\frac{1}{2}}$ denote the respective projections onto these spaces.

We can twist the Dirac operator D_0 with the cotangent bundle to define the twisted Dirac operator $D_1 \colon \Gamma(T^*M \otimes \mathbb{S}) \to \Gamma(T^*M \otimes \mathbb{S})$ given by $D_1 = (\mathrm{id} \otimes \mu) \circ \nabla^S$, which acts locally on decomposable elements as

$$D_1(X \otimes \phi) = X \otimes (D_0 \phi) + \sum_i (\nabla_i X) \otimes [e_i \cdot \phi]. \tag{2.5}$$

One may alternatively define this operator as the Dirac operator D_1 on $T^*M \otimes \mathbb{S}$ with the induced structure given by

$$W \cdot (X \otimes \phi) = X \otimes (W \cdot \phi) \tag{2.6}$$

and

$$\nabla^S_W (X \otimes \phi) = (\nabla_W X) \otimes \phi + X \otimes (\nabla^S_W \phi). \tag{2.7}$$

Indeed, we have

$$\begin{aligned} \sum_i e_i \cdot \nabla^S_i (X \otimes \phi) &= \sum_i e_i \cdot \Big[(\nabla_i X) \otimes \phi + X \otimes (\nabla^S_i \phi) \Big] \\ &= \sum_i (\nabla_i X) \otimes [e_i \cdot \phi] + X \otimes \Big[\sum_i e_i \cdot (\nabla^S_i \phi) \Big] \\ &= \sum_i (\nabla_i X) \otimes [e_i \cdot \phi] + X \otimes (D_0 \phi). \end{aligned} \tag{2.8}$$

With respect to the decomposition (2.4), we may write D_1 in the block matrix form

$$D_1 = \begin{bmatrix} \frac{2-n}{n} \iota \circ D_0 \circ \mu & 2\iota \circ P^* \\ \frac{2}{n} P \circ \mu & Q \end{bmatrix}, \tag{2.9}$$

where $P = \mathrm{pr}_{\frac{3}{2}} \circ \nabla^S$ is Penrose operator (see [HS19, Wan91]). The operator Q is called the Rarita–Schwinger operator, which is a self-adjoint first order differential operator.

Definition 2.1. A Rarita–Schwinger field is a section of $\mathbb{S}_{\frac{3}{2}}$ which is in the kernel of D_1.

These fields were first considered in [RS41] and have since been studied extensively in physics. Recently, Rarita–Schwinger fields have been a topic of growing focus in the mathematics literature [BM21, HS19, HT20, OT21, Wan91].

3 Manifolds with G_2 Structure

We now review some basics of G_2 geometry. For a more in-depth introduction see [Kar20, Sua20].

Definition 3.1. A 3-form on a 7-dimensional manifold M is called positive if for any non-zero $Y \in T_pM$,

$$(Y \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi \neq 0. \tag{3.1}$$

A positive 3-form is also called a G_2 structure and induces a unique Riemannian metric g and associated volume form vol by

$$-\frac{1}{6}(Y \lrcorner \varphi) \wedge (Z \lrcorner \varphi) \wedge \varphi = g(Y, Z)\mathrm{vol}. \tag{3.2}$$

Remark 3.2. Some authors may omit the minus sign in (3.2), which results in flipping the orientation of the metric.

The metric and volume form in turn induce a Hodge star operator $\star$ and a dual 4-form $\psi = \star\varphi$.

We have the following well-known characterization of 7-manifolds which admit G_2 structures which we present without proof:

Proposition 3.3. *A 7-manifold M admits a G_2 structure if and only if it is orientable and spinnable.*

A G_2 structure φ also defines a vector cross-product $\times$ which acts by

$$X \times Y = (Y \lrcorner [X \lrcorner \varphi])^\sharp. \tag{3.3}$$

There are several important contraction identities which we will employ throughout these notes, their derivations can be found in [Kar09].

Proposition 3.4. *On a manifold M with G_2 structure φ, the tensors g, φ and ψ satisfy the following identities:*

- *Contractions of φ with φ:*

$$\begin{aligned}\varphi_{ijk}\varphi_{pqk} &= g_{ip}g_{jq} - g_{iq}g_{jp} - \psi_{ijpq},\\ \varphi_{ijk}\varphi_{pjk} &= g_{ip},\\ \varphi_{ijk}\varphi_{ijk} &= 42;\end{aligned} \tag{3.4}$$

- *Contractions of ψ with ψ:*

$$\begin{aligned}\psi_{ijkl}\psi_{pqkl} &= 4g_{ip}g_{jq} - 4g_{iq}g_{jp} - 2\psi_{ijpq},\\ \psi_{ijkl}\psi_{pjkl} &= 24g_{ip},\\ \psi_{ijkl}\psi_{ijkl} &= 168;\end{aligned} \tag{3.5}$$

- *Contractions of φ with ψ:*

$$\begin{aligned}\varphi_{ijk}\psi_{pqrk} &= g_{ip}\varphi_{jqr} + g_{iq}\varphi_{pjr} + g_{ir}\varphi_{pqj} - g_{jp}\varphi_{iqr} - g_{jq}\varphi_{pir} - g_{jr}\varphi_{pqi},\\ \varphi_{ijk}\psi_{pqjk} &= -4\varphi_{ipq},\\ \varphi_{ijk}\psi_{pijk} &= 0.\end{aligned} \tag{3.6}$$

3.1 Decomposition of Forms and the $\diamond$ Operator

On a manifold M with G_2 structure, the bundle $\Lambda(T^*M) = \oplus_{k=1}^{7}\Lambda^k(T^*M)$ decomposes fibrewise into irreducible representations of the group G_2. This allows us to decompose the spaces Ω^k of k-forms.

Proposition 3.5. *On a manifold M with G_2 structure φ, the spaces Ω^2 and Ω^3 of 2- and 3-forms respectively can be orthogonally decomposed into irreducible G_2 representations. In particular, we have*

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}, \tag{3.7}$$

and

$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}. \tag{3.8}$$

Each space Ω^k_ℓ has pointwise dimension ℓ and can be described invariantly as follows:

$$\Omega^2_7 = \{X \lrcorner \varphi \mid X \in \mathfrak{X}\}, \tag{3.9}$$

$$\Omega^2_{14} = \{\beta \in \Omega^2 \mid \beta \wedge \psi = 0\}, \tag{3.10}$$

$$\Omega^3_1 = \{f\varphi \mid f \in \Omega^0\}, \tag{3.11}$$

$$\Omega^3_7 = \{X \lrcorner \psi \mid X \in \mathfrak{X}\}, \tag{3.12}$$

$$\Omega^3_{27} = \{\gamma \in \Omega^3 \mid \gamma \wedge \varphi = \gamma \wedge \psi = 0\}. \tag{3.13}$$

We may also decompose the spaces of 4- and 5-forms. These can be obtained by applying the Hodge star to those above.

We have a couple of identities regarding contractions of 2-forms and the 4-form ψ.

Proposition 3.6. *Let $\beta_7 \in \Omega^2_7$ and $\beta_{14} \in \Omega^2_{14}$ be 2-forms on a manifold M with G_2 structure φ. Then*

$$(\beta_7)_{pq}\psi_{ijpq} = -4(\beta_7)_{ij}, \tag{3.14}$$

$$(\beta_{14})_{pq}\psi_{ijpq} = 2(\beta_{14})_{ij}. \tag{3.15}$$

Using the decomposition (3.7), we obtain a full decomposition of the space $\mathcal{T}^2$ of 2-tensors. In particular, we have

$$\mathcal{T}^2 = \Omega^0 \cdot g \oplus \mathcal{S}^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}, \tag{3.16}$$

where $\mathcal{S}^2_0$ denotes the space of traceless symmetric 2-tensors.

We can define a linear map from the space $\mathcal{T}^2$ of 2-tensors to the space of 3-forms. Given a 2-tensor C, we let $C \diamond \varphi$ be the 3-form given locally by

$$(C \diamond \varphi)_{ijk} = C_{ip}\varphi_{pjk} - C_{jp}\varphi_{pik} + C_{kp}\varphi_{pij}. \tag{3.17}$$

It can be shown that the map $C \mapsto C \diamond \varphi$ is a surjective linear map and its kernel is precisely the space Ω^2_{14} and that we have the alternative characterization of the spaces Ω^3_1, Ω^3_7, and Ω^3_{27}, that is,

$$\Omega^3_1 = \{C \diamond \varphi \mid C \in \Omega^0 \cdot g\}, \tag{3.18}$$

$$\Omega^3_7 = \{C \diamond \varphi \mid C \in \Omega^2_7\}, \tag{3.19}$$

$$\Omega^3_{27} = \{C \diamond \varphi \mid C \in \mathcal{S}^2_0\}. \tag{3.20}$$

We have the following result involving the Hodge star acting on forms in Ω^3_{27}.

Proposition 3.7. *Let $C \in \mathcal{S}^2_0$ be a traceless symmetric 2-tensor on a manifold M with G_2 structure φ. If $\gamma = C \diamond \varphi$, then*

$$(\star\gamma)_{ijkl} = -C_{ip}\psi_{pjkl} + C_{jp}\psi_{pikl} - C_{kp}\psi_{pijl} + C_{lp}\psi_{pijk}. \tag{3.21}$$

3.2 Torsion and Curvature

We recall that a G_2 structure φ induces a Riemannian metric g. We can thus consider its Levi-Civita connection ∇. An important observation about the Levi-Civita covariant derivative of a G_2 structure is the following:

Proposition 3.8. *Let $X \in \mathfrak{X}$ be a vector field on a manifold M with G_2 structure φ. Then the 3-form $\nabla_X\varphi$ lies in Ω^3_7.*

Each form in Ω^3_7 can be written as $Y \lrcorner \psi$ for some vector field Y. This motivates the definition of the torsion tensor.

Definition 3.9. Let $X \in \mathfrak{X}$ be a vector field on a manifold M with G_2 structure φ. We can write

$$\nabla_X\varphi = T(X) \lrcorner \psi$$

for some vector field $T(X)$ on M. It follows that there exists a 2-tensor T, called the torsion tensor, such that

$$\nabla_l\varphi_{ijk} = T_{lp}\psi_{pijk}. \tag{3.22}$$

There is an alternative characterization of torsion using decomposition of forms.

Definition 3.10. The torsion forms of a G_2 structure φ are

$$\tau_0 \in \Omega^0, \qquad \tau_1 \in \Omega^1, \qquad \tau_2 \in \Omega^2_{14}, \qquad \tau_3 \in \Omega^3_{27} \tag{3.23}$$

which are defined by

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + \star\tau_3, \tag{3.24}$$

$$d\psi = 4\tau_1 \wedge \psi + \star\tau_2. \tag{3.25}$$

The torsion tensor and torsion forms are related by

$$T = \frac{1}{4}\tau_0 g - \tau_3' + \tau_1' - \frac{1}{2}\tau_2 \tag{3.26}$$

where $\tau_1' \in \Omega^2_7$ and $\tau_3' \in \mathcal{S}^2_0$ with $\tau_1 = (\tau_1')^\sharp \lrcorner \varphi$ and $\tau_3 = \tau_3' \diamond \varphi$.

We obtain 16 classes of G_2 structure depending on which components of torsion are non-zero. A few important classes are the following:

Definition 3.11. A G_2 structure φ is said to be:

- closed if $d\varphi = 0$ (or equivalently $\tau_0 = \tau_1 = \tau_3 = 0$),
- coclosed if $d\psi = 0$ (or equivalently $\tau_1 = \tau_2 = 0$),

- nearly parallel if $\tau_1 = \tau_2 = \tau_3 = 0$ (so $d\varphi = \tau_0\psi$ and $d\psi = 0$),
- torsion-free if $d\varphi = d\psi = 0$.

There is a relation between the torsion of a G_2 structure and its curvature. One identity which demonstrates this is the so-called "G_2 Bianchi identity" from [Kar09].

Proposition 3.12. *On a manifold M with G_2 structure φ, we have*

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ip}T_{jq}\varphi_{pqk} + \frac{1}{2}\mathrm{Rm}_{ijpq}\varphi_{pqk}. \tag{3.27}$$

We also have the following lemma.

Lemma 3.13. *On a manifold M with G_2 structure φ, we have*

$$\mathrm{Rm}_{ipqr}\psi_{jpqr} = 0. \tag{3.28}$$

Using the Lemma, we can contract the G_2 Bianchi identity to obtain a result regarding the Ricci tensor.

Proposition 3.14. *On a manifold M with G_2 structure φ, we have*

$$\mathrm{Ric}_{ij} = (\nabla_p T_{iq} - \nabla_i T_{pq})\varphi_{pqj} - T_{ip}T_{pj} + (\operatorname{tr} T)T_{ij} + T_{kp}T_{iq}\psi_{jkpq}, \tag{3.29}$$

3.3 Nearly Parallel G_2 Structures

We now discuss nearly parallel G_2 structures. We keep this section short by restricting to properties that we will use in the sequel. Further research on nearly parallel G_2 structures can be found in [AS12, DS20, NS21, SWW22].

As we have seen, a G_2 structure φ is nearly parallel if the only non-zero component of its torsion is T_1. Alternatively, we have $d\varphi = \tau_0\psi$ and $d\psi = 0$. When the manifold M is connected, we obtain a condition on the function τ_1.

Proposition 3.15. *Let M be a connected manifold with nearly parallel G_2 structure φ. Then τ_1 is constant.*

Proof. Applying the exterior derivative to $d\varphi = \tau_0\psi$ and using $d\psi = 0$ yields

$$0 = d\tau_0 \wedge \psi + \tau_0 d\psi = d\tau_0 \wedge \psi. \tag{3.30}$$

The map $\alpha \mapsto \alpha \wedge \psi$ defines an isomorphism between Ω^1 and Ω^5_7, hence $d\tau_0 = 0$. Since M is connected, τ_0 must be constant. □

For simplicity, we will henceforth assume that our manifold M is connected.

Using the relation (3.26) between the torsion forms and torsion tensor, we have

$$T = \frac{1}{4}\tau_0 g. \tag{3.31}$$

Since τ_0 is constant, T is parallel. An application of Proposition 3.14 shows that

$$\mathrm{Ric}_{ij} = \frac{3}{8}\tau_0^2 g_{ij}, \tag{3.32}$$

and hence a nearly parallel G_2 structure yields an Einstein metric. It then follows that the scalar curvature R is given by $\frac{21}{8}\tau_0^2$.

When our G_2 structure φ is nearly parallel, we obtain nice contractions of the Riemann curvature tensor and the forms φ and ψ.

Proposition 3.16. *On a manifold M with a nearly parallel G_2 structure, we have*

$$\frac{1}{2}\mathrm{Rm}_{ijpq}\varphi_{pqk} = -\frac{1}{16}\tau_0^2\varphi_{ijk}, \tag{3.33}$$

$$\frac{1}{2}R_{ijpq}\psi_{pqkl} = \mathrm{Rm}_{ijkl} + \frac{1}{16}\tau_0^2\Big(g_{ik}g_{jl} - g_{il}g_{jk} - \psi_{ijkl}\Big). \tag{3.34}$$

Proof. The first identity can be obtained from the G_2 Bianchi identity (3.27) by plugging in $T = \frac{1}{4}\tau_0 g$. The second identity follows from the first and the first identity in (3.4). □

An important property of nearly parallel G_2 structures is that they admit a real Killing spinor, that is a spinor ϕ and a constant c satisfying

$$\nabla^S_X \phi = cX \cdot \phi. \tag{3.35}$$

In our case, it can be shown that the constant $c = \frac{\tau_0}{8}$ (see [Bär93, BFGK90]).

4 Dirac Bundle Structures

4.1 The Spinor Bundle

We recall that any manifold M admitting a G_2 structure is spinnable. We also have that the spinor bundle $\mathbb{S}$ of a 7-manifold is a rank 8 real vector bundle. When M is a manifold with G_2 structure, there is an identification of $\mathbb{S}$ with the octonion bundle $\mathbb{O} = \underline{\mathbb{R}} \oplus TM$. The latter bundle is nice to work with since sections on it consist of a function and a vector field. This is done in [Kar10, Gri17] by identifying $\mathbb{S}$ with the octonions. As such, we inherit a Dirac bundle structure on $\mathbb{O}$. A concrete description of the bundle isomorphism is given in Section 8 of [Gri17]. We review this structure below and as an abuse of notation, we will use the terms spinors and octionions interchangeably. Additionally, we will identify the bundles $\mathbb{S}$ and $\mathbb{O}$ with one another.

Let Y be a vector field on M and let (f, Z) be a spinor. The Clifford multiplication on this bundle is given by octonionic multiplication:

$$Y \cdot (f,\, Z) = (0,\, Y) \cdot (f,\, Z) = (-\langle Y, Z\rangle,\, fY + Y \times Z). \tag{4.1}$$

In coordinates, this is given by

$$Y \cdot (f,\, Z) = (-Y_i Z_i,\, fY_a + Y_i Z_j \varphi_{ija}). \tag{4.2}$$

If X is another vector field on M, we can check that the Clifford identity holds:

$$X \cdot (Y \cdot (f,\, Z)) + Y \cdot (X \cdot (f,\, Z)) = -2\langle X, Y\rangle(f,\, Z). \tag{4.3}$$

Further, skew-adjointness of the Clifford multiplication holds: Let (h, W) be another spinor, then

$$\langle Y \cdot (f,\, Z), (h,\, W)\rangle + \langle (f,\, Z), Y \cdot (h,\, W)\rangle = 0. \tag{4.4}$$

It has been shown that the spin connection ∇^S under the bundle isomorphism is given by the Levi-Civita connection and the torsion tensor. In particular, we have the following equation:

$$\begin{aligned}\nabla^S_X(f,\, Z) &= (\nabla_X f,\, \nabla_X Z) + (f, Z) \cdot \Big(0,\, \frac{1}{2} X \lrcorner T\Big) \\ &= (\nabla_X f,\, \nabla_X Z) + \Big(-\frac{1}{2}\langle Z, X \lrcorner T\rangle,\, \frac{1}{2} f(X \lrcorner T) + \frac{1}{2} Z \times (X \lrcorner T)\Big).\end{aligned} \tag{4.5}$$

Routine calculations show that this operator satisfies the Leibniz rule and that in the special case where φ is a torsion-free G_2 structure, the spin connection ∇^S is just the Levi-Civita connection ∇.

One can also verify that this connection is compatible with the induced metric on $\underline{\mathbb{R}} \oplus TM$ and also with the Levi-Civita connection (see Appendix A).

4.2 The Bundle of Spinor-Valued 1-Forms

We now extend these results to the bundle $T^*M \otimes \mathbb{S}$. The Dirac bundle structure on this bundle is induced by that of the one on $\mathbb{S}$. Let W, X be 1-forms and let (f, Z) be a spinor. The induced Clifford multiplication on this bundle is given by

$$W \cdot [X \otimes (f,\, Z)] = X \otimes (W \cdot (f,\, Z)) \tag{4.6}$$

extended linearly.

The connection (which we will again denote ∇^S) on this bundle is induced by the Levi-Civita connection on M as well as the spin connection ∇^S on $\mathbb{S}$. That is,

$$\nabla^S_W(X \otimes (f,\, Z)) = \nabla_W X \otimes (f,\, Z) + X \otimes \nabla^S_W(f,\, Z). \tag{4.7}$$

By the linearity of Clifford multiplication as well as the connection, we may instead write these in terms of a 1-form Y and a 2-tensor C. Define the right action of $\mathbb{S}$ on a pair (Y, C) in coordinates by

$$(Y,\, C)\cdot(f,\, Z) = (fY_a - C_{ai}Z_i,\; fC_{ab} + Y_aZ_b + C_{ai}Z_j\varphi_{ijb}). \tag{4.8}$$

Similarly, the Clifford multiplication on the pair (Y, C) by the 1-form X can be written in coordinates by

$$X\cdot(Y,\, C) = (-X_iC_{ai},\; X_bY_a + X_iC_{aj}\varphi_{ijb}). \tag{4.9}$$

These are just the regular octonion and Clifford multiplications on the "right-most" $(0, 1)$ indices of (Y, C). We then have

$$\nabla^S_X(Y,\, C) = (\nabla_X Y,\, \nabla_X C) + (Y,\, C)\cdot\Big(0,\, \frac{1}{2}X \lrcorner T\Big). \tag{4.10}$$

The fact that ∇^S is a metric connection and that it is compatible with the Levi-Civita connection can be done similarly to the $\mathbb{S}$ case which gives us the Dirac bundle structure.

5 The Twisted Dirac Operator

We want to compute the action of the Dirac operator D_1 on this bundle. In local coordinates, we have

$$\begin{aligned}
D_1(Y, C) &= \sum_i e_i\cdot[\nabla^S_i(Y, C)]\\
&= \sum_i e_i\cdot\Big[(\nabla_iY,\, \nabla_iC) + \Big(-\frac{1}{2}\langle C, e_i\lrcorner T\rangle,\, \frac{1}{2}Y\otimes(e_i\lrcorner T) + \frac{1}{2}C\times(e_i\lrcorner T)\Big)\Big]\\
&= \sum_i(-\langle\nabla_iC, e_i\rangle,\, (\nabla_iY)\otimes e_i + e_i\times(\nabla_iC))\\
&\quad + \Big(-\frac{1}{2}Y\langle e_i, e_i\lrcorner T\rangle - \frac{1}{2}\langle e_i, C\times(e_i\lrcorner T)\rangle,\, -\frac{1}{2}\langle C, e_i\lrcorner T\rangle\otimes e_i + \frac{1}{2}Y\otimes(e_i\times(e_i\lrcorner T)) + \frac{1}{2}e_i\times(C\times(e_i\lrcorner T))\Big)\\
&= (-(\nabla_iC_{ai}),\, (\nabla_bY_a) + (\nabla_kC_{al})\varphi_{klb})\\
&\quad + \Big(-\frac{1}{2}T_{ii}Y_a - \frac{1}{2}C_{ak}T_{il}\varphi_{kli},\, -\frac{1}{2}C_{ai}T_{bi} + \frac{1}{2}Y_aT_{ik}\varphi_{ikb} + \frac{1}{2}C_{ak}T_{il}\varphi_{klj}\varphi_{ijb}\Big)\\
&= (-(\nabla_iC_{ai}),\, (\nabla_bY_a) + (\nabla_kC_{al})\varphi_{klb})\\
&\quad + \Big(-\frac{1}{2}T_{ii}Y_a - \frac{1}{2}C_{ak}T_{il}\varphi_{kli},\, -\frac{1}{2}C_{ai}T_{bi} + \frac{1}{2}Y_aT_{ik}\varphi_{ikb} + \frac{1}{2}C_{ab}T_{ii} - \frac{1}{2}C_{ai}T_{ib} - \frac{1}{2}C_{ak}T_{il}\psi_{klbi}\Big)
\end{aligned} \tag{5.1}$$

Using the decomposition of the torsion tensor (3.26) from before, this may be rewritten as

$$\begin{aligned}
D_1(Y, C) &= (-(\nabla_iC_{ai}),\, (\nabla_bY_a) + (\nabla_kC_{al})\varphi_{klb})\\
&\quad + \Big(-\frac{7}{8}\tau_0Y_a + 3C_{ai}(\tau_1)_i,\, 3Y_a(\tau_1)_b + \frac{5}{8}\tau_0C_{ab} + 2C_{ai}(\tau_1')_{ib} + \frac{1}{2}C_{ai}(\tau_2)_{ib} + C_{ai}(\tau_3')_{ib}\Big)
\end{aligned} \tag{5.2}$$

5.1 The Nearly Parallel Case

Restricting our attention to the nearly parallel G_2 case, we recall that $\tau_1 = \tau_2 = \tau_3 = 0$. As such, we may write the Dirac operator D_1 as

$$D_1(Y, C) = (-\operatorname{div} C,\, \operatorname{grad} Y + \operatorname{curl} C) + \Big(-\frac{7}{8}\tau_0Y,\, \frac{5}{8}\tau_0C\Big). \tag{5.3}$$

Here div, grad, and curl are extensions of the usual first-order operators. They act in coordinates by

$$(\operatorname{div} C)_a = \nabla_iC_{ai}, \qquad (\operatorname{grad} Y)_{ab} = \nabla_bY_a, \qquad (\operatorname{curl} C)_{ab} = (\nabla_kC_{al})\varphi_{klb}. \tag{5.4}$$

Similarly to the previous section, we can check that

$$\begin{aligned}
D_1^2(Y, C) &= D_1(-\operatorname{div} C,\, \operatorname{grad} Y + \operatorname{curl} C) + D_1\Big(-\frac{7}{8}\tau_0Y,\, \frac{5}{8}\tau_0C\Big)\\
&= (-\operatorname{div}(\operatorname{grad} Y) - \operatorname{div}(\operatorname{curl} C),\, -\operatorname{grad}(\operatorname{div} C) + \operatorname{curl}(\operatorname{grad} Y) + \operatorname{curl}(\operatorname{curl} C))\\
&\quad + \Big(\frac{7}{8}\tau_0(\operatorname{div} C),\, \frac{5}{8}\tau_0(\operatorname{grad} Y) + \frac{5}{8}\tau_0(\operatorname{curl} C)\Big) + \Big(-\frac{5}{8}\tau_0(\operatorname{div} C),\, -\frac{7}{8}\tau_0(\operatorname{grad} Y) + \frac{5}{8}\tau_0(\operatorname{curl} C)\Big)\\
&\quad + \Big(\frac{49}{64}\tau_0^2Y,\, \frac{25}{64}\tau_0^2C\Big).
\end{aligned} \tag{5.5}$$

We have the following identities for the second-order terms appreaing in the above equation:

Proposition 5.1. *On a manifold M with nearly parallel G_2 structure φ, we have*

$$[\operatorname{curl}(\operatorname{grad} Y)]_{ab} = -\frac{1}{16}\tau_0^2 Y_p \varphi_{pab}, \tag{5.6}$$

$$-[\operatorname{div}(\operatorname{curl} C)]_a = -\frac{1}{16}\tau_0^2 C_{pq}\varphi_{pqa}, \tag{5.7}$$

$$-[\operatorname{div}(\operatorname{grad} Y)]_a = (\Delta Y)_a, \tag{5.8}$$

$$\begin{aligned}[\operatorname{curl}(\operatorname{curl} C)]_{ab} - [\operatorname{grad}(\operatorname{div} C)]_{ab} &= (\Delta C)_{ab} - 2\mathrm{Rm}_{pabq}C_{pq} - \tau_0(\operatorname{curl} C)_{ab} \\ &\quad + \frac{3}{8}\tau_0^2 C_{ab} - \frac{1}{16}\tau_0^2 C_{ba} + \frac{1}{16}\tau_0^2 C_{pp}g_{ab} + \frac{1}{16}\tau_0^2 C_{pq}\psi_{pqab}.\end{aligned} \tag{5.9}$$

Proof. We prove these identities in coordinates. First, we have

$$\begin{aligned}\operatorname{curl}(\operatorname{grad} Y) &= (\nabla_i(\operatorname{grad} Y)_{aj})\varphi_{ijb} = (\nabla_i\nabla_j Y_a)\varphi_{ijb} = \frac{1}{2}[(\nabla_i\nabla_j - \nabla_j\nabla_i)Y_a]\varphi_{ijb} \\ &= -\frac{1}{2}\mathrm{Rm}_{ijap}Y_p\varphi_{ijb} = \frac{1}{16}\tau_0^2 Y_p\varphi_{apb} = -\frac{1}{16}\tau_0^2 Y_p\varphi_{pab}.\end{aligned} \tag{5.10}$$

Next, we check

$$\begin{aligned}-[\operatorname{div}(\operatorname{curl} C)]_a &= -\nabla_k(\operatorname{curl} C)_{ak} = -\nabla_k[(\nabla_i C_{aj})\varphi_{ijk}] = -(\nabla_k\nabla_i C_{aj})\varphi_{ijk} - (\nabla_i C_{aj})(\nabla_k\varphi_{ijk}) \\ &= -\frac{1}{2}[(\nabla_k\nabla_i - \nabla_i\nabla_k)C_{aj}]\varphi_{ijk} + \frac{1}{4}\tau_0(\nabla_i C_{aj})\psi_{kijk} = \frac{1}{2}(\mathrm{Rm}_{kiap}C_{pj} + \mathrm{Rm}_{kijp}C_{ap})\varphi_{ijk} \\ &= -\frac{1}{16}\tau_0^2 C_{pj}\varphi_{apj} - \frac{1}{16}\tau_0^2 C_{ap}\varphi_{jpj} \\ &= -\frac{1}{16}\tau_0^2 C_{pq}\varphi_{pqa}.\end{aligned} \tag{5.11}$$

We also have

$$-[\operatorname{div}(\operatorname{grad} Y)]_a = -\nabla_k(\operatorname{grad} Y)_{ak} = -\nabla_k\nabla_k Y_a = (\Delta Y)_a. \tag{5.12}$$

Lastly,

$$\begin{aligned}&[\operatorname{curl}(\operatorname{curl} C)]_{ab} - [\operatorname{grad}(\operatorname{div} C)]_{ab} \\ &= (\nabla_i(\operatorname{curl} C)_{aj})\varphi_{ijb} - \nabla_b(\operatorname{div} C)_a \\ &= \nabla_i[(\nabla_k C_{al})\varphi_{klj}]\varphi_{ijb} - (\nabla_b\nabla_i C_{ai}) \\ &= (\nabla_i\nabla_k C_{al})\varphi_{klj}\varphi_{ijb} + (\nabla_k C_{al})(\nabla_i\varphi_{klj})\varphi_{ijb} - (\nabla_b\nabla_i C_{ai}) \\ &= (\nabla_i\nabla_b C_{ai}) - (\nabla_i\nabla_i C_{ab}) - (\nabla_i\nabla_k C_{al})\psi_{klbi} + \frac{1}{4}\tau_0(\nabla_k C_{al})\psi_{iklj}\varphi_{ijb} - (\nabla_b\nabla_i C_{ai}) \\ &= -(\nabla_i\nabla_i C_{ab}) + [(\nabla_i\nabla_b - \nabla_b\nabla_i)C_{ai}] - \frac{1}{2}[(\nabla_i\nabla_k - \nabla_k\nabla_i)C_{al}]\psi_{klbi} - \tau_0(\nabla_k C_{al})\varphi_{klb} \\ &= -(\nabla_i\nabla_i C_{ab}) - (\mathrm{Rm}_{ibap}C_{pi} + \mathrm{Rm}_{ibip}C_{ap}) \\ &\qquad + \frac{1}{2}(\mathrm{Rm}_{ikap}C_{pl} + \mathrm{Rm}_{iklp}C_{ap})\psi_{klbi} - \tau_0(\nabla_k C_{al})\varphi_{klb} \\ &= -(\nabla_i\nabla_i C_{ab}) - \mathrm{Rm}_{pabi}C_{pi} + \mathrm{Ric}_{bp}C_{ap} \\ &\qquad + \Big[\mathrm{Rm}_{ambl} + \frac{1}{16}\tau_0^2(g_{ab}g_{ml} - g_{al}g_{mb} - \psi_{ambl})\Big]C_{ml} - \tau_0(\nabla_k C_{al})\varphi_{klb} \\ &= -(\nabla_i\nabla_i C_{ab}) - \mathrm{Rm}_{pabq}C_{pq} + \mathrm{Ric}_{bp}C_{ap} \\ &\qquad - \mathrm{Rm}_{mabl}C_{ml} + \frac{1}{16}\tau_0^2 C_{ll}g_{ab} - \frac{1}{16}\tau_0^2 C_{ba} + \frac{1}{16}\tau_0^2\psi_{abml}C_{ml} - \tau_0(\nabla_k C_{al})\varphi_{klb} \\ &= (\Delta C)_{ab} - 2\mathrm{Rm}_{pabq}C_{pq} - \tau_0(\operatorname{curl} C)_{ab} \\ &\qquad + \frac{3}{8}\tau_0^2 C_{ab} - \frac{1}{16}\tau_0^2 C_{ba} + \frac{1}{16}\tau_0^2 C_{pp}g_{ab} + \frac{1}{16}\tau_0^2 C_{pq}\psi_{pqab}\end{aligned} \tag{5.13}$$

□

As a result, we may write

$$D_1^2(Y,C) = ((\Delta Y)_a, (\Delta C)_{ab} - 2\mathrm{Rm}_{pabq}C_{pq}) + \frac{1}{4}\tau_0((\operatorname{div} C)_a, -(\operatorname{grad} Y)_{ab} + (\operatorname{curl} C)_{ab}) \\ + \frac{49}{64}\tau_0^2(Y_a, C_{ab}) + \frac{1}{16}\tau_0^2(-C_{pq}\varphi_{pqa},\, -Y_p\varphi_{pab} + C_{pp}g_{ab} - C_{ba} + C_{pq}\psi_{pqab}). \tag{5.14}$$

5.2 Laplacian Operators

We now discuss the properties of certain Laplacian operators acting on 2-tensors. Given a 2-tensor C, we may decompose it into a symmetric component $C^+ \in \Omega^0 \cdot g \oplus \mathcal{S}_0^2$ and an antisymmetric component $C^- \in \Omega_7^2 \oplus \Omega_{14}^2$. On symmetric 2-tensors, we have the Lichnerowicz Laplacian Δ_L, which acts on C^+ by

$$(\Delta_L C^+)_{ab} = (\Delta C^+)_{ab} + \mathrm{Ric}_{ap}C^+_{pb} + \mathrm{Ric}_{bp}C^+_{ap} - 2\mathrm{Rm}_{pabq}C^s_{pq} = (\Delta C^+)_{ab} + \frac{3}{4}\tau_0^2 C^+_{ab} - 2\mathrm{Rm}_{pabq}C^+_{pq}. \tag{5.15}$$

On the other hand, we have the Hodge Laplacian on 2-forms, which by direct computation acts in a similar fashion on C^- by

$$\begin{aligned}(\Delta_d C^-)_{ab} &= (dd^*C^-)_{ab} + (d^*dC^-)_{ab} \\ &= \nabla_a(d^*C^-)_b - \nabla_b(d^*C^-)_a - \nabla_p(dC^-)_{pab} \\ &= \nabla_a(-\nabla_p C^-_{pb}) - \nabla_b(-\nabla_p C^-_{pa}) - \nabla_p(\nabla_p C^-_{ab} - \nabla_a C^-_{pb} + \nabla_b C^-_{pa}) \\ &= -\nabla_p\nabla_p C^-_{ab} + [(\nabla_p\nabla_a - \nabla_a\nabla_p)C^-_{pb}] - [(\nabla_p\nabla_b - \nabla_b\nabla_p)]C^-_{pa} \\ &= (\Delta C^-)_{ab} - \mathrm{Rm}_{papm}C^-_{mb} - \mathrm{Rm}_{pabm}C^-_{pm} + \mathrm{Rm}_{pbpm}C_{ma} + \mathrm{Rm}_{pbam}C_{pm} \\ &= (\Delta C^-)_{ab} + \mathrm{Ric}_{am}C^-_{mb} - \mathrm{Rm}_{pabm}C^-_{pm} - \mathrm{Ric}_{bm}C_{ma} - \mathrm{Rm}_{mabp}C_{mp} \\ &= (\Delta C^-)_{ab} + \frac{3}{4}\tau_0^2 C^-_{ab} - 2\mathrm{Rm}_{pabq}C^-_{pq}.\end{aligned} \tag{5.16}$$

For simplicity, we use $\Delta_{d/L}$ to denote the operator on a 2-tensor which acts as the Lichnerowicz Laplacian on its symmetric part and the Hodge Laplacian on its antisymmetric part.

Recalling that

$$\begin{aligned}(\Delta_d Y)_a &= (dd^*Y)_a + (d^*dY)_a \\ &= \nabla_a(d^*Y) - \nabla_p(dY)_{pa} \\ &= \nabla_a(-\nabla_p Y_p) - \nabla_p(\nabla_p Y_a - \nabla_a Y_p) \\ &= -\nabla_p\nabla_p Y_a + [(\nabla_p\nabla_a - \nabla_a\nabla_p)Y_p] \\ &= (\Delta Y)_a - \mathrm{Rm}_{papm}Y_m \\ &= (\Delta Y)_a + \mathrm{Ric}_{am}Y_m \\ &= (\Delta Y)_a + \frac{3}{8}\tau_0^2 Y_a,\end{aligned} \tag{5.17}$$

we may then write

$$D_1^2(Y,C) = ((\Delta_d Y)_a,\, (\Delta_{d/L}C)_{ab}) + \frac{1}{4}\tau_0((\operatorname{div} C)_a,\, -(\operatorname{grad} Y)_{ab} + (\operatorname{curl} C)_{ab}) \\ + \tau_0^2\Big(\frac{25}{64}Y_a,\, \frac{1}{64}C_{ab}\Big) + \frac{1}{16}\tau_0^2(-C_{pq}\varphi_{pqa},\, -Y_p\varphi_{pab} + C_{pp}g_{ab} - C_{ba} + C_{pq}\psi_{pqab}). \tag{5.18}$$

Before we compute the Rarita–Schwinger fields, we first note how the Lichnerowicz Laplacian splits with respect to the decomposition $\mathcal{S}^2 = \Omega^0 \cdot g \oplus \mathcal{S}_0^2$.

We may write any symmetric 2-tensor C^+ as

$$C^+ = fg + C_{27} \tag{5.19}$$

where $f \in \Omega^0$, and $C_{27} \in \mathcal{S}_0^2$. We compute how the Lichnerowicz Laplacians interact with each component of C.

First, direct computations show that

$$\begin{aligned}(\Delta_L(fg))_{ab} &= -\nabla_p\nabla_p(fg_{ab}) + \frac{3}{4}\tau_0^2(fg_{ab}) - 2\mathrm{Rm}_{iabj}(fg_{ij}) \\ &= -(\nabla_p\nabla_p f)g_{ab} + \frac{3}{4}f\tau_0^2 g_{ab} - 2f\mathrm{Ric}_{ab} \\ &= (\Delta f)g_{ab}.\end{aligned}$$

It then follows that $\Delta_L(fg) = (\Delta f)g$ and so $\Delta_L(fg) \in \Omega^0 \cdot g$.

Next, we can check that

$$\begin{aligned}(\Delta_L C_{27})_{ab} &= -\nabla_p\nabla_p(C_{27})_{ab} + \frac{3}{4}\tau_0^2(C_{27})_{ab} - 2\mathrm{Rm}_{iabj}(C_{27})_{ij} \\ &= -\nabla_p\nabla_p(C_{27})_{ba} + \frac{3}{4}\tau_0^2(C_{27})_{ba} - 2\mathrm{Rm}_{jbai}(C_{27})_{ji} \\ &= (\Delta_L C_{27})_{ba}\end{aligned}$$

and

$$\begin{aligned}(\Delta_L C_{27})_{ii} &= -\nabla_p\nabla_p(C_{27})_{ii} + \frac{3}{4}\tau_0^2(C_{27})_{ii} - 2\mathrm{Rm}_{kiil}(C_{27})_{kl} \\ &= -2\mathrm{Ric}_{kl}(C_{27})_{kl} \\ &= -\frac{3}{4}\tau_0^2 g_{kl}(C_{27})_{kl} \\ &= 0.\end{aligned}$$

Thus $\Delta_L C_{27}$ is both symmetric and traceless and so $\Delta_L C_{27} \in \mathcal{S}^2_0$. From this we see that Δ_L splits with respect to the decomposition $\mathcal{S}^2 = \Omega^0 \cdot g \oplus \mathcal{S}^2_0$ of symmetric tensors.

5.3 $\frac{1}{2}$- and $\frac{3}{2}$-Spinors

We recall the decomposition $T^*M \otimes \mathbb{S} = \mathbb{S}_{\frac{1}{2}} \oplus \mathbb{S}_{\frac{3}{2}}$.

Using the projection maps, we can compute the spaces $\mathbb{S}_{\frac{1}{2}}$ and $\mathbb{S}_{\frac{3}{2}}$ on a manifold with G_2 structure.

$$\begin{aligned}\mathrm{pr}_{\frac{1}{2}}(Y, C) &= \iota \circ \mu(Y, C) \\ &= \iota\Big(\sum_i e_i \cdot (Y_i, C_{ib})\Big) \\ &= \iota(-C_{ii}, Y_a + C_{kl}\varphi_{klb}) \\ &= -\frac{1}{7}\sum_j e_j \otimes [e_j \cdot (-C_{ii}, Y_a + C_{kl}\varphi_{klb})] \\ &= -\frac{1}{7}(-Y_a - C_{kl}\varphi_{kla}, -C_{kk}g_{ab} - Y_l\varphi_{lab} + C_{ba} - C_{ab} + C_{kl}\psi_{klab}) \\ &= \frac{1}{7}(Y_a + C_{kl}\varphi_{kla}, C_{kk}g_{ab} + Y_l\varphi_{lab} + C_{ab} - C_{ba} - C_{kl}\psi_{klab})\end{aligned}$$

Using the decomposition $\mathcal{T}^2 = \Omega^0 \cdot g \oplus \mathcal{S}^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}$, we can write $C = fg + C_{27} + Z \lrcorner \varphi + C_{14}$. By considering the appropriate terms, we can see that

$$\mathrm{pr}_{\frac{1}{2}}(Y, fg + C_{27} + Z \lrcorner \varphi + C_{14}) = \Big(\frac{1}{7}Y + \frac{6}{7}Z, fg + \Big[\frac{1}{7}Y + \frac{6}{7}Z\Big] \lrcorner \varphi\Big). \tag{5.20}$$

Taking the complement, we also see that

$$\mathrm{pr}_{\frac{3}{2}}(Y, fg + C_{27} + Z \lrcorner \varphi + C_{14}) = \Big(\frac{6}{7}Y - \frac{6}{7}Z, C_{27} + \Big[\frac{6}{7}Y - \frac{6}{7}Z\Big] \lrcorner \varphi + C_{14}\Big). \tag{5.21}$$

It follows that $\mathbb{S}_{\frac{1}{2}}$ consists of elements of the form

$$(Y, fg + Y \lrcorner \varphi) \tag{5.22}$$

and $\mathbb{S}_{\frac{3}{2}}$ consists of elements of the form

$$\Big(Y, C_{27} - \frac{1}{6}Y \lrcorner \varphi + C_{14}\Big). \tag{5.23}$$

5.4 Rarita–Schwinger Fields

We now compute the space of Rarita–Schwinger fields.

Theorem 5.2. *On a compact manifold M with nearly parallel G_2 structure φ, the space of Rarita–Schwinger fields is isomorphic to the space*

$$\Big\{C \in \mathcal{S}^2_0 \mid \operatorname{div} C = 0,\, \operatorname{curl} C = -\frac{5}{8}\tau_0 C\Big\} = \Big\{C \in \mathcal{S}^2_0 \mid \operatorname{div} C = 0,\, \Delta_L C = \frac{13}{64}\tau_0^2 C\Big\}. \tag{5.24}$$

Proof. Since D_1 is self-adjoint and M is compact, we have

$$D_1(Y,\, C) = 0 \iff D_1^2(Y,\, C) = 0. \tag{5.25}$$

Our previous discussion shows that

$$D_1(Y,\, C) = 0 \iff \begin{cases} -(\nabla_i C_{ai}) - \frac{7}{8}\tau_0 Y_a = 0, \\ (\nabla_b Y_a) + (\nabla_k C_{al})\varphi_{klb} + \frac{5}{8}\tau_0 C_{ab} = 0. \end{cases} \tag{5.26}$$

and that

$$D_1^2(Y,\, C) = 0 \iff \begin{cases} (\Delta_d Y)_a + \frac{1}{4}\tau_0(\nabla_i C_{ai}) + \frac{25}{64}\tau_0^2 Y_a - \frac{1}{16}\tau_0^2 C_{pq}\varphi_{pqa} = 0, \\ (\Delta_{d/L} C)_{ab} - \frac{1}{4}\tau_0(\nabla_b Y_a) + \frac{1}{4}\tau_0(\nabla_i C_{aj})\varphi_{ijb} \\ \quad + \frac{1}{64}\tau_0^2 C_{ab} - \frac{1}{16}\tau_0^2 Y_p \varphi_{pab} + \frac{1}{16}\tau_0^2 C_{pp} g_{ab} - \frac{1}{16}\tau_0^2 C_{ba} + \frac{1}{16}\tau_0^2 C_{pq}\psi_{pqab} = 0. \end{cases} \tag{5.27}$$

Applying $-\nabla_b$ to the second equation in (5.26), we get

$$\begin{aligned} 0 &= -\nabla_b\nabla_b Y_a - \nabla_b[(\nabla_k C_{al})\varphi_{klb}] - \frac{5}{8}\tau_0 \nabla_b C_{ab} \\ &= -\nabla_b\nabla_b Y_a - (\nabla_b\nabla_k C_{al})\varphi_{klb} - (\nabla_k C_{al})(\nabla_b \varphi_{klb}) - \frac{5}{8}\tau_0\nabla_b C_{ab} \\ &= -\nabla_b\nabla_b Y_a - \frac{1}{2}[(\nabla_b\nabla_k - \nabla_k\nabla_b)C_{al}]\varphi_{klb} - \frac{1}{4}\tau_0(\nabla_k C_{al})\psi_{bklb} - \frac{5}{8}\tau_0\nabla_b C_{ab} \\ &= -\nabla_b\nabla_b Y_a + \frac{1}{2}(\mathrm{Rm}_{bkam} C_{ml}\varphi_{klb} + \mathrm{Rm}_{bklm} C_{am}\varphi_{klb}) - \frac{5}{8}\tau_0\nabla_b C_{ab} \\ &= -\nabla_b\nabla_b Y_a - \frac{1}{16}\tau_0^2 C_{ml}\varphi_{aml} - \frac{5}{8}\tau_0\nabla_b C_{ab}. \end{aligned}$$

Substituting in the first equation, we get

$$0 = -\nabla_b\nabla_b Y_a + \frac{35}{64}\tau_0^2 Y_a - \frac{1}{16}\tau_0^2 C_{ml}\varphi_{mla} = (\Delta_d Y)_a + \frac{11}{64}\tau_0^2 Y_a - \frac{1}{16}\tau_0^2 C_{ml}\varphi_{mla}.$$

We recall that since we are restricting our attention to $\mathbb{S}_{\frac{3}{2}}$, the spinor-valued 1-form $(Y,\, C)$ is of the form

$$(Y,\, C) = \Big(Y,\, C_{27} - \frac{1}{6} Y \lrcorner \varphi + C_{14}\Big).$$

It follows that

$$C_{ml}\varphi_{mla} = -Y_a.$$

Plugging this into our earlier result, we get that

$$(\Delta_d Y)_a + \frac{15}{64} Y_a = 0,$$

and since the Hodge Laplacian has no negative eigenvalues, we must have $Y = 0$.

This reduces the second equation in (5.26) to

$$(\nabla_k C_{al})\varphi_{klb} = -\frac{5}{8}\tau_0 C_{ab}. \tag{5.28}$$

We can plug this and $Y = 0$ into the second equation of (5.27) results in

$$(\Delta_{d/L}C)_{ab} - \frac{9}{64}\tau_0^2 C_{ab} - \frac{1}{16}\tau_0^2 C_{ba} + \frac{1}{16}\tau_0^2 C_{kk} g_{ab} + \frac{1}{16}\tau_0^2 C_{ij}\psi_{iabj} = 0.$$

Now using the decomposition $C = C_{27} + C_{14}$, we get an equation for C_{27} and another for C_{14}:

$$(\Delta_L C_{27})_{ab} - \frac{13}{64}\tau_0^2 (C_{27})_{ab} + (\Delta_d C_{14})_{ab} + \frac{3}{64}\tau_0^2 (C_{14})_{ab} = 0. \tag{5.29}$$

The terms involving C_{27} all lie in the space $\mathcal{S}_0^2$ and those involving C_{14} all lie in the space Ω^2. As these spaces are orthogonal, we may consider two separate equations

$$\begin{aligned}(\Delta_L C_{27})_{ab} - \frac{13}{64}\tau_0^2 (C_{27})_{ab} &= 0\\ (\Delta_d C_{14})_{ab} + \frac{3}{64}\tau_0^2 (C_{14})_{ab} &= 0.\end{aligned} \tag{5.30}$$

As before, the Hodge Laplacian has no negative eigenvalues, and so $C_{14} = 0$. Plugging this in, we see that the space of Rarita–Schwinger fields is isomorphic to

$$\left\{C \in \mathcal{S}_0^2 \mid \operatorname{div} C = 0,\ \operatorname{curl} C = -\frac{5}{8}\tau_0 C\right\} = \left\{C \in \mathcal{S}_0^2 \mid \operatorname{div} C = 0,\ \Delta_L C = \frac{13}{64}\tau_0^2 C\right\} \tag{5.31}$$

as desired.

□

Remark 5.3. The same equations can be used to find Rarita–Schwinger fields in the case where φ is a torsion-free G_2 structure. Here, $\tau_0 = 0$ so we can conclude that Y must be a harmonic 1-form, leaving only the equation $\Delta_{d/L}C = 0$.

When φ is torsion-free, both the Hodge and Lichnerowicz Laplacians split with respect to our decompositions and so $\Delta_d C_{14} = 0$ and $\Delta_L C_{27} = 0$. Finally, one can show that

$$\Delta_L C_{27} = 0 \iff \Delta_d(C_{27} \diamond \varphi) = 0 \tag{5.32}$$

hence the space of Rarita–Schwinger fields in the torsion-free case is isomorphic to

$$\mathcal{H}^1 \oplus \mathcal{H}^2_{14} \oplus \mathcal{H}^3_{27}, \tag{5.33}$$

which has dimension equal to $b_1 + b^2_{14} + b^3_{27}$. This result was originally proven by [Wan91].

We recall that an Einstein metric g on a manifold M is called linearly unstable if there exists a non-zero 2-tensor that is transverse traceless, that is,

$$\operatorname{div} h = 0 \text{ and } \operatorname{tr} h = 0, \tag{5.34}$$

such that

$$\int_M \langle (\Delta_L - 2\lambda)h, h\rangle < 0, \tag{5.35}$$

where λ denotes the Einstein constant. The metric g is called stable otherwise. If g were a stable Einstein metric, we can see that the Lichnerowicz Laplacian cannot have eigenvalues less than 2λ when restricted to transeverse traceless tensors.

The conditions from the previous theorem tell us that C must be transverse traceless. Since $\frac{13}{64}\tau_0^2 < \frac{3}{4}\tau_0^2 = 2\lambda$, we obtain the following corollary.

Corollary 5.4. *If M is a manifold with nearly parallel G_2 structure φ and g is a stable metric, then the space of Rarita–Schwinger fields is trivial.*

We end this with yet another characterization of the space of Rarita–Schwinger fields, matching that of [Ohn22].

Corollary 5.5. *On a compact manifold M with nearly parallel G_2 structure φ, the space of Rarita–Schwinger fields is isomorphic to the space*

$$\left\{\gamma \in \Omega^3_{27} \mid \star d\gamma = -\frac{1}{8}\tau_0\gamma\right\}. \tag{5.36}$$

Proof. Recall that we have an isomorphism between the spaces $\mathcal{S}^2_0$ and Ω^3_{27} given by the $\diamond$ operator. If $C \in \mathcal{S}^2_0$, then its corresponding $\gamma = C \diamond \varphi$ is determined by

$$\gamma_{ijk} = C_{ip}\varphi_{pjk} - C_{jp}\varphi_{pik} + C_{kp}\varphi_{pij}. \tag{5.37}$$

By (3.21), we have

$$(\star\gamma)_{ijkl} = -C_{iq}\psi_{qjkl} + C_{jq}\psi_{qikl} - C_{kq}\psi_{qijl} + C_{lq}\psi_{qijk}. \tag{5.38}$$

We can compute that

$$\begin{aligned}
(\star d\gamma)_{ijk} &= (d^* \star \gamma)_{ijk} \\
&= -\nabla_p(\star\gamma)_{pijk} \\
&= \nabla_p(C_{pq}\psi_{qijk}) - \nabla_p(C_{iq}\psi_{qpjk}) + \nabla_p(C_{jq}\psi_{qpik}) - \nabla_p(C_{kq}\psi_{pqij}) \\
&= (\nabla_p C_{pq})\psi_{qijk} + C_{pq}(\nabla_p\psi_{qijk}) - (\nabla_p C_{iq})\psi_{qpjk} - C_{iq}(\nabla_p\psi_{qpjk}) \\
&\quad\; + (\nabla_p C_{jq})\psi_{qpik} + C_{jq}(\nabla_p\psi_{qpik}) - (\nabla_p C_{kq})\psi_{pqij} - C_{kp}(\nabla_p\psi_{pqij}) \\
&= (\nabla_p C_{pq})\psi_{qijk} - (\nabla_p C_{iq})\psi_{qpjk} + (\nabla_p C_{jq})\psi_{qpik} - (\nabla_p C_{kq})\psi_{pqij} \\
&\quad\; - \frac{1}{4}\tau_0 C_{pq}\Big(g_{pq}\varphi_{ijk} - g_{pi}\varphi_{qjk} + g_{pj}\varphi_{qik} - g_{pk}\varphi_{qij}\Big) \\
&\quad\; + \frac{1}{4}\tau_0 C_{iq}\Big(g_{pq}\varphi_{pjk} - g_{pp}\varphi_{qjk} + g_{pj}\varphi_{qpk} - g_{pk}\varphi_{qpj}\Big) \\
&\quad\; - \frac{1}{4}\tau_0 C_{jq}\Big(g_{pq}\varphi_{pik} - g_{pp}\varphi_{qik} + g_{pi}\varphi_{qpk} - g_{pk}\varphi_{qpi}\Big) \\
&\quad\; + \frac{1}{4}\tau_0 C_{kq}\Big(g_{pq}\varphi_{pij} - g_{pp}\varphi_{qij} + g_{pi}\varphi_{qpj} - g_{pj}\varphi_{qpi}\Big) \\
&= (\nabla_p C_{pq})\psi_{qijk} - (\nabla_p C_{iq})\psi_{qpjk} + (\nabla_p C_{jq})\psi_{qpik} - (\nabla_p C_{kq})\psi_{pqij} \\
&\quad\; - \frac{3}{4}\tau_0\Big(C_{iq}\varphi_{qjk} - C_{jq}\varphi_{qik} + C_{kq}\varphi_{qij}\Big) \\
&= (\nabla_p C_{pq})\psi_{qijk} - (\nabla_p C_{iq})\psi_{qpjk} + (\nabla_p C_{jq})\psi_{qpik} - (\nabla_p C_{kq})\psi_{pqij} - \frac{3}{4}\tau_0\gamma_{ijk}.
\end{aligned} \tag{5.39}$$

From Theorem 5.2 the space of Rarita–Schwinger fields is isomorphic to the space

$$\Big\{C \in \mathcal{S}^2_0 \mid \operatorname{div} C = 0,\ \operatorname{curl} C = -\frac{5}{8}\tau_0 C\Big\} = \Big\{C \in \mathcal{S}^2_0 \mid \operatorname{div} C = 0,\ \Delta_L C = \frac{13}{64}\tau_0^2 C\Big\}. \tag{5.40}$$

Suppose C is an element of the above space. Since $\operatorname{curl} C = -\frac{5}{8}\tau_0 C$, we have

$$\begin{aligned}
&- (\nabla_p C_{iq})\psi_{qpjk} + (\nabla_p C_{jq})\psi_{qpik} - (\nabla_p C_{kq})\psi_{pqij} \\
&= -(\nabla_p C_{iq})\Big(\varphi_{qpl}\varphi_{jkl} - g_{qj}g_{pk} + g_{qk}g_{pj}\Big) + (\nabla_p C_{jq})\Big(\varphi_{qpl}\varphi_{ikl} - g_{qi}g_{pk} + g_{qk}g_{pi}\Big) \\
&\quad\; - (\nabla_p C_{kq})\Big(\varphi_{qpl}\varphi_{ijl} - g_{qi}g_{pj} + g_{qj}g_{pi}\Big) \\
&= \frac{5}{8}\tau_0 C_{il}\varphi_{jkl} + \nabla_k C_{ij} - \nabla_j C_{ik} - \frac{5}{8}\tau_0 C_{jl}\varphi_{ikl} - \nabla_k C_{ji} + \nabla_i C_{jk} \\
&\quad\; + \frac{5}{8}\tau_0 C_{kl}\varphi_{ijl} + \nabla_j C_{ki} - \nabla_i C_{kj} \\
&= \frac{5}{8}\tau_0\Big(C_{iq}\varphi_{qjk} - C_{jq}\varphi_{qik} + C_{kq}\varphi_{qij}\Big) \\
&= \frac{5}{8}\tau_0\gamma_{ijk}.
\end{aligned} \tag{5.41}$$

Further, since $\operatorname{div} C = 0$, we have

$$(\nabla_p C_{pq})\psi_{qijk} = 0. \tag{5.42}$$

Combining these together, we get

$$(\star d\gamma)_{ijk} = -\frac{1}{8}\tau_0\gamma_{ijk}, \tag{5.43}$$

as desired.

Conversely, suppose that γ satisfies $\star d\gamma = -\frac{1}{8}\tau_0\gamma$. We can again check that

$$\begin{aligned}
&-(\nabla_p C_{iq})\psi_{qpjk} + (\nabla_p C_{jq})\psi_{qpik} - (\nabla_p C_{kq})\psi_{pqij} \\
&= -(\nabla_p C_{iq})\Big(\varphi_{qpl}\varphi_{jkl} - g_{qj}g_{pk} + g_{qk}g_{pj}\Big) + (\nabla_p C_{jq})\Big(\varphi_{qpl}\varphi_{ikl} - g_{qi}g_{pk} + g_{qk}g_{pi}\Big) \\
&\quad - (\nabla_p C_{kq})\Big(\varphi_{qpl}\varphi_{ijl} - g_{qi}g_{pj} + g_{qj}g_{pi}\Big) \\
&= (\nabla_p C_{iq})\varphi_{pql}\varphi_{ljk} + \nabla_k C_{ij} - \nabla_j C_{ik} - (\nabla_p C_{jq})\varphi_{pql}\varphi_{lik} - \nabla_k C_{ji} + \nabla_i C_{jk} \\
&\quad + (\nabla_p C_{kq})\varphi_{pql}\varphi_{lij} + \nabla_j C_{ki} - \nabla_i C_{kj} \\
&= (\nabla_p C_{iq})\varphi_{pql}\varphi_{ljk} - (\nabla_p C_{jq})\varphi_{pql}\varphi_{lik} + (\nabla_p C_{kq})\varphi_{pql}\varphi_{lij} \\
&= (\operatorname{curl} C \diamond \varphi)_{ijk}.
\end{aligned} \tag{5.44}$$

As such

$$-\frac{1}{8}\tau_0\gamma_{ijk} = -(\nabla p C_{pq})\psi_{qijk} + (\operatorname{curl} C \diamond \varphi)_{ijk} - \frac{3}{4}\tau_0\gamma_{ijk}. \tag{5.45}$$

We can check that $\operatorname{curl} C$ lies entirely in $\mathcal{S}^2_0$. First, we have

$$(\operatorname{curl} C)_{ii} = (\nabla_p C_{iq})\varphi_{pqi} = \nabla_p(C_{iq}\varphi_{pqi}) - C_{iq}(\nabla_p\varphi_{pqi}) = -\frac{1}{4}\tau_0 C_{iq}\psi_{ppqi} = 0, \tag{5.46}$$

and so $\operatorname{curl} C$ is traceless.

Contract $\operatorname{curl} C$ on both indices with φ, we can see that its associated vector field is

$$\begin{aligned}
(\operatorname{curl} C)_{ij}\varphi_{ijk} &= (\nabla_p C_{iq})\varphi_{pqj}\varphi_{ijk} \\
&= (\nabla_p C_{iq})\Big(g_{pk}g_{qi} - g_{pi}g_{qk} - \psi_{pqki}\Big) \\
&= (\nabla_k C_{qq}) - (\nabla_p C_{pk}) - (\nabla_p C_{iq})\psi_{pqki} \\
&= -(\nabla_p C_{pk}) - \nabla_p(C_{iq}\psi_{pqki}) + C_{iq}(\nabla_p\psi_{pqki}) \\
&= -(\nabla_p C_{pk}) - \frac{\tau_0}{4}C_{iq}\Big(g_{pp}\varphi_{qki} - g_{pq}\varphi_{pki} + g_{pk}\varphi_{pqi} - g_{pi}\varphi_{pqk}\Big) \\
&= -(\nabla_p C_{pk}).
\end{aligned} \tag{5.47}$$

Hence its Ω^2_7 component is

$$[\pi_7(\operatorname{curl} C)]_{kl} = -\frac{1}{6}(\nabla_p C_{pq})\varphi_{qkl}. \tag{5.48}$$

The equation $\star d\gamma = -\frac{1}{8}\tau_0\gamma$ lies entirely in Ω^3_{27}. Since

$$-\frac{1}{6}\Big((\operatorname{div} C) \lrcorner \varphi\Big) \diamond \varphi = \frac{1}{2}(\operatorname{div} C) \lrcorner \psi, \tag{5.49}$$

we must have $\operatorname{div} C = 0$.

We also note that our assumption regarding γ implies that it is coclosed since

$$d^*\gamma = -8\tau_0^{-1}d^* \star d\gamma = 8\tau_0^{-1} \star d \star \star d\gamma = 8\tau_0^{-1} \star d^2\gamma = 0. \tag{5.50}$$

We can check that

$$\begin{aligned}
(d^*\gamma)_{ij} &= -\nabla_p\gamma_{pij} \\
&= -\nabla_p\Big(C_{pq}\varphi_{qij} - C_{iq}\varphi_{qpj} + C_{jq}\varphi_{qpi}\Big) \\
&= -(\nabla_p C_{pq})\varphi_{qij} - C_{pq}(\nabla_p\varphi_{qij}) + (\nabla_p C_{iq})\varphi_{qpj} + C_{iq}(\nabla_p\varphi_{qpj}) - (\nabla_p C_{jq})\varphi_{qpi} - C_{jq}(\nabla_p\varphi_{qpi}) \\
&= \frac{1}{4}\tau_0 C_{pq}\psi_{pqij} - (\operatorname{curl} C)_{ij} - \frac{1}{4}\tau_0\psi_{pqpj} + (\operatorname{curl} C)_{ji} + \frac{1}{4}\tau_0 C_{jq}\psi_{pqpi} \\
&= -2[\pi_{14}(\operatorname{curl} C)]_{ij},
\end{aligned} \tag{5.51}$$

and so $\operatorname{curl} C$ lies entirely in $\mathcal{S}^2_0$.

Using (5.39), we have that

$$-\frac{1}{8}\tau_0 C \diamond \varphi = \Big((\operatorname{curl} C) - \frac{3}{4}\tau_0 C\Big) \diamond \varphi. \tag{5.52}$$

We then get an equation in $\mathcal{S}^2_0$

$$-\frac{1}{8}\tau_0 C = \operatorname{curl} C - \frac{3}{4}\tau_0 C, \tag{5.53}$$

and we can conclude that $\operatorname{curl} C = -\frac{5}{8}\tau_0 C$ as desired.

□

A Compatibility of the Spin Connection

In this appendix, we verify the compatibility of the spin connection ∇^S on the octonion bundle $\mathbb{O}$.

We check that ∇^S is a metric connection and that it is compatible with the Levi-Civita connection ∇. First, to see that ∇^S is a metric connection, we compute:

$$\begin{aligned}
\nabla_X\langle(f,Z),(h,W)\rangle &= \nabla_X(fh + \langle Z,W\rangle) \\
&= (\nabla_X f)h + f(\nabla_X h) + \langle \nabla_X Z, W\rangle + \langle Z, \nabla_X W\rangle \\
&= X_k(\nabla_k f)h + fX_k(\nabla_k h) + X_k(\nabla_k Z_i)W_i + Z_iX_k(\nabla_k W_i).
\end{aligned}$$

$$\begin{aligned}
\langle \nabla^S_X(f,Z),(h,W)\rangle &= \langle(\nabla_X f, \nabla_X Z),(h,W)\rangle \\
&\quad + \Big\langle\Big(-\frac{1}{2}\langle Z, X \lrcorner T\rangle, \frac{1}{2}f(X \lrcorner T) + \frac{1}{2}Z \times (X \lrcorner T)\Big),(h,W)\Big\rangle \\
&= (\nabla_X f)h + \langle \nabla_X Z, W\rangle \\
&\quad - \frac{1}{2}\langle Z, X \lrcorner T\rangle h + \frac{1}{2}f\langle X \lrcorner T, W\rangle + \frac{1}{2}\langle Z \times (X \lrcorner T), W\rangle \\
&= X_k(\nabla_k f)h + X_k(\nabla_k Z_i)W_i - \frac{1}{2}Z_iX_pT_{pi}h + \frac{1}{2}fX_pT_{pi}W_i + \frac{1}{2}Z_kX_pT_{pl}\varphi_{kli}W_i.
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\langle (f,Z),\nabla^S_X(h,W)\rangle &= \langle(f,Z),(\nabla_X h, \nabla_X W)\rangle \\
&\quad + \Big\langle(f,Z),\Big(-\frac{1}{2}\langle W, X \lrcorner T\rangle, \frac{1}{2}h(X \lrcorner T) + \frac{1}{2}W \times (X \lrcorner T)\Big)\Big\rangle \\
&= f(\nabla_X h) + \langle Z, \nabla_X W\rangle \\
&\quad - \frac{1}{2}f\langle W, X \lrcorner T\rangle + \frac{1}{2}h\langle Z, X \lrcorner T\rangle + \frac{1}{2}\langle Z, W \times (X \lrcorner T)\rangle \\
&= fX_k(\nabla_k h) + Z_iX_k(\nabla_k W_i) - \frac{1}{2}fW_iX_pT_{pi} + \frac{1}{2}hZ_iX_pT_{pi} + \frac{1}{2}Z_iW_kX_pT_{pl}\varphi_{kli}.
\end{aligned}$$

Comparing the equations above yields

$$\nabla_X\langle(f,Z),(h,W)\rangle = \langle \nabla^S_X(f,Z),(h,W)\rangle + \langle(f,Z),\nabla^S_X(h,W)\rangle \tag{A.2}$$

as desired.

Next, we check compatibility with the Levi-Civita connection:

$$
\begin{aligned}
\nabla^S_X[Y\cdot(f,Z)] &= \nabla^S_X(-\langle Y,Z\rangle, fY+Y\times Z))\\
&= (-\nabla_X\langle Y,Z\rangle, \nabla_X(fY)+\nabla_X(Y\times Z))\\
&\quad+\Big(-\frac{1}{2}f\langle Y,X\lrcorner T\rangle-\frac{1}{2}\langle Y\times Z,X\lrcorner T\rangle,-\frac{1}{2}\langle Y,Z\rangle(X\lrcorner T)+\frac{1}{2}fY\times(X\lrcorner T)+\frac{1}{2}(Y\times Z)\times(X\lrcorner T)\Big)\\
&= \Big(-X_k(\nabla_kY_i)Z_i-Y_iX_k(\nabla_kZ_i), X_k(\nabla_kf)Y_a+fX_k(\nabla_kY_a)+X_k(\nabla_kY_i)Z_j\varphi_{ija}+Y_iX_k(\nabla_kZ_j)\varphi_{ija}+Y_iZ_jX_k(\nabla\\
&\quad+\Big(-\frac{1}{2}fY_iX_pT_{pi}-\frac{1}{2}Y_kZ_l\varphi_{kli}X_pT_{pi},-\frac{1}{2}Y_iZ_iX_pT_{pa}+\frac{1}{2}fY_iX_pT_{pj}\varphi_{ija}+\frac{1}{2}Y_kZ_l\varphi_{kli}X_pT_{pj}\varphi_{ija}\Big)\\
&= \Big(-X_k(\nabla_kY_i)Z_i-Y_iX_k(\nabla_kZ_i)-\frac{1}{2}fY_iX_pT_{pi}-\frac{1}{2}Y_kZ_l\varphi_{kli}X_pT_{pi},\\
&\qquad X_k(\nabla_kf)Y_a+fX_k(\nabla_kY_a)+X_k(\nabla_kY_i)Z_j\varphi_{ija}+Y_iX_k(\nabla_kZ_j)\varphi_{ija}+Y_iZ_jX_kT_{kp}\psi_{pija}\\
&\qquad-\frac{1}{2}Y_iZ_iX_pT_{pa}+\frac{1}{2}fY_iX_pT_{pj}\varphi_{pja}+\frac{1}{2}Y_kZ_aX_pT_{pk}-\frac{1}{2}Y_aT_kX_pT_{pk}-\frac{1}{2}Y_kZ_lX_pT_{pj}\psi_{klja}\Big)\\
&= \Big(-X_k(\nabla_kY_i)Z_i-Y_iX_k(\nabla_kZ_i)-\frac{1}{2}fY_iX_pT_{pi}-\frac{1}{2}Y_kZ_l\varphi_{kli}X_pT_{pi},\\
&\qquad X_k(\nabla_kf)Y_a+fX_k(\nabla_kY_a)+X_k(\nabla_kY_i)Z_j\varphi_{ija}+Y_iX_k(\nabla_kZ_j)\varphi_{ija}+\frac{1}{2}Y_iZ_kX_kT_{kp}\psi_{pija}\\
&\qquad-\frac{1}{2}Y_iZ_iX_pT_{pa}+\frac{1}{2}fY_iX_pT_{pj}\varphi_{pja}+\frac{1}{2}Y_kZ_aX_pT_{pk}-\frac{1}{2}Y_aT_kX_pT_{pk}\Big).
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_XY)\cdot(f,Z) &= (-\langle\nabla_XY,Z\rangle, f(\nabla_XY)+(\nabla_XY)\times Z)\\
&= \Big(-X_k(\nabla_kY_i)Z_i, fX_k(\nabla_kY_a)+X_k(\nabla_kY_i)Z_j\varphi_{ija}\Big).
\end{aligned}
$$

$$
\begin{aligned}
Y\cdot[\nabla^S_X(f,Z)] &= Y\cdot\Big[(\nabla_Xf,\nabla_XZ)+\Big(-\frac{1}{2}\langle Z,X\lrcorner T\rangle,\frac{1}{2}f(X\lrcorner T)+\frac{1}{2}Z\times(X\lrcorner T)\Big)\Big]\\
&= (-\langle Y,\nabla_XZ\rangle,(\nabla_Xf)Y+Y\times(\nabla_XZ))\\
&\quad+\Big(-\frac{1}{2}f\langle Y,X\lrcorner T\rangle-\frac{1}{2}\langle Y,Z\times(X\lrcorner T)\rangle,-\frac{1}{2}\langle Z,X\lrcorner T\rangle Y+\frac{1}{2}fY\times(X\lrcorner T)+\frac{1}{2}Y\times(Z\times(X\lrcorner T))\Big)\\
&= \Big(-Y_iX_k(\nabla_kZ_i), X_k(\nabla_kf)Y_a+Y_iX_k(\nabla_kZ_j)\varphi_{ija}\Big)\\
&\quad+\Big(-\frac{1}{2}Y_iX_pT_{pi}-\frac{1}{2}Y_iZ_kX_pT_{pl}\varphi_{kli},-\frac{1}{2}Z_iX_pT_{pi}Y_a+\frac{1}{2}fY_iX_pT_{pj}\varphi_{ija}+\frac{1}{2}Y_iZ_kX_pT_{pl}\varphi_{klj}\varphi_{ija}\Big)\\
&= \Big(-Y_iX_k(\nabla_kZ_i)-\frac{1}{2}Y_iX_pT_{pi}-\frac{1}{2}Y_iZ_kX_pT_{pl}\varphi_{kli},\\
&\qquad X_k(\nabla_kf)Y_a+Y_iX_k(\nabla_kZ_j)\varphi_{ija}-\frac{1}{2}Z_iX_pT_{pi}Y_a+\frac{1}{2}fY_iX_pT_{pj}\varphi_{ija}\\
&\qquad+\frac{1}{2}Y_iZ_aX_pT_{pi}-\frac{1}{2}Y_iZ_iX_pT_{pa}-\frac{1}{2}Y_iZ_kX_pT_{pl}\psi_{klai}\Big).
\end{aligned}
\tag{A.3}
$$

Again, by comparing the equations, we conclude that

$$
\nabla^S_X[Y\cdot(f,Z)] = (\nabla_XY)\cdot(f,Z)+Y\cdot[\nabla^S_X(f,Z)] \tag{A.4}
$$

and so ∇^S is compatible with ∇.

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